Research statement

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During my predoctoral studies I was interested in a couple of informal questions which belong to both computation theory and dynamical systems theory :

1. the one of the difference that exists between the model of the Turing machine - which according to the intuition captures the very notion of computation (Church-Turing thesis) - and the actual computation done by the human brain
2. and the one of the possibility to define a quantity of 'organisedness' for dynamical systems which would capture the difference, in terms of complexity, between actual living beings and the machines that human construct.

The motivation for this last question lies in particular in the unability of quantities such as Topological entropy or Kolmogorov complexity and derived ones to capture the intuitive 'complexity' of what we *experience* as complex. I believe that the idea that they underlie is more related to randomness (see for instance the definition of Martin-Löf randomness).

My intuition on the first question was that the difference between Turing machines and the brain lie in constraints, in terms of dynamics, that are constantly applied on the brain and to which it has to adapt, explaining the complexity of its description compared to the one of Turing machines. This led me to study, with M. Sablik during my doctoral thesis and then with N.Aubrun and M.Rao during my first year of post-doc, some apparented questions about the effect of some dynamical constraints on the expressive power of a class of computation models, namely multidimensional subshifts of finite type. I will describe this research work in the Section 1 of this text, and related research program in Section 2.

After this first post-doc I learnt about G.Tononi's theory of consciousness, namely Integrated information theory, which, based on causality formalism developped by J.Pearl[Pearl], aims at a quantification of the degree of consciousness of physical systems. Since G.Tononi's ideas were similar to the ones I developed during my reflexion of the second question, I chose to join G.Tononi's team for a second post-doc, motivated also by the attention that other known mathematicians in my research field gave to his work. Despite the fact that I had a very critical look at the precise formulation of G.Tononi's theory, this interaction was very fruitful. In particular I realized that adapting causality formalism to the framework of multidimensional subshifts of finite type could lead in the long run to an answer to this second question, as well as insights in questions related to information transfer in (symbolic) dynamical systems. I will explain my work in this direction in Section 3 as well as related research directions.

More recently while comparing Integrated information theory and the 'strong AI' thesis (according to which, roughly, one can understand the humain mind by analogy with artificial intelligence) defended in particular by D.Dennett, and working as a researcher in the field of machine learning, I got interested naturally in the widely considered question about the efficiency of machine learning with neural networks. I also explain how the tools I developped during my second post-doctoral year can lead to important insights into this question.

I would also like to notice that, despite its apparent heterogeneous character, this whole research statement is meant to be limited to few general research areas that are strongly interconnected : each progress made in one of these areas, with contact with other researchers along their development, can have important impact in the others.

1 Frontiers of computability in symbolic dynamics

1.1 Multidimensional subshifts of finite type

Multidimensional subshifts of finite type are dynamical systems that consist in the action of the shift on a set of colorings of the grid $\mathbb{Z}^n$ ($n \geq 2$) with a finite set of colors and defined by a finite set of forbidden patterns. They appeared in various different contexts : first in the work of C. Shannon [Shannon] as models for discrete communication channels, as discretisations of continuous dynamical systems (for instance in Rudolph's contribution to the $\times 2 \times 3$ conjecture [R90], or more recently the study of surface diffeomorphisms [BCS18]), as statistical physics models, in particular exactly solvable lattice models [Baxter] [K61] [L67] and recently their connections with percolation theory and phase transitions (for instance [D15]), and in constructions of plane tilings involved in undecidability results [B66] [Rob71].

They are defined formally as follows :

Définition 1. *A multidimensional subshift of finite type (often abbreviated* ***SFT****) on the alphabet* $\mathcal{A}$ *is a subset* X *of* $\mathcal{A}^{\mathbb{Z}^k}$*, width* $k \geq 2$*, such that there exists some finite set of patterns* $\mathcal{F}$ *such that the elements of* X *(called configurations) are the ones in which no element of* $\mathcal{F}$ *appears.*

In this definition, a pattern is an element of some $\mathcal{A}^{\mathbb{U}}$, where $\mathbb{U}$ is a finite subset of $\mathbb{Z}^k$. Let us call shift action the action of $\mathbb{Z}^k$ on $\mathcal{A}^{\mathbb{Z}^k}$ such that a vector $\mathbf{u}$ acts on a configuration $x \in \mathcal{A}^{\mathbb{Z}^k}$ to output the configuration $(x_{\mathbf{u}+\mathbf{v}})_{\mathbf{v} \in \mathcal{A}^{\mathbb{Z}^k}}$.

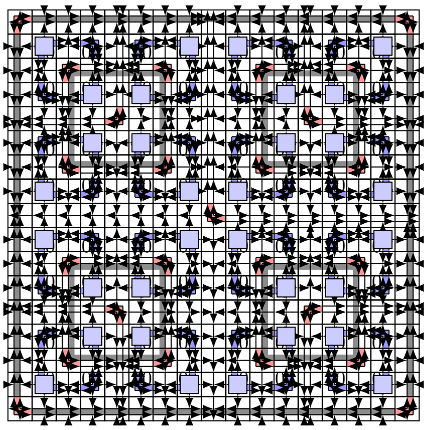
A multidimensional subshift of finite type, together with the shift action of $\mathbb{Z}^k$ (restricted on the subshift) is a dynamical system. In the following, we consider mainly bidimensional ones, meaning that $k = 2$.

1.2 Recent construction methods

Recently, the techniques involved in R.Robinson's construction [Rob71] (of tilings in which are implemented computations of arbitrary Turing machines via the inclusion of arbitrarily large space-time diagrams of these machines) have been used by M.Hochman and T.Meyerovitch to show that one can give precise answers to questions about the dynamics of multidimensional subshifts of finite type using tools coming from computability theory. In particular :

1. The characterization of the possible values of entropy for these dynamical systems [HM10] as well
2. A similar characterization of the values of entropy dimension [M11] (another topological invariant that was defined in order to study zero entropy dynamical systems),
3. The simulation of effective dynamical systems with tridimensional subshifts of finite type
4. The characterization of entropies of cellular automata in dimension greater than three [H09].

To give an idea of how Turing computations are embedded in a bidimensional subshift of finite type, I represented in the following figure a typical pattern in Robinson subshift, used as a basis for these constructions. In this pattern, the blue squares that are strictly contained in a gray square all together are the support of the (finite) space-time diagram of a machine.



Using this schema, M. Hochman et T. Meyerovitch proved in particular the following theorem :

Théorème 1 (Hochman, Meyerovitch)**.** *The possible values of entropy for multidimensional subshifts of finite type are exactly the non-negative* Π_1*-computable real numbers.*

For a bidimensional subshift X, the entropy is expressed as follows :

$$h(X) = \lim_n \frac{\log_2(N_n(X)))}{n^2},$$

where $N_n(X)$ is the number of size n square patterns that appear in a configuration of X. Moreover, a number x is said to be Π_1-computable when there exists an algorithm which, given as input an integer n, outputs a rational number r_n such that the sequence (r_n) is non-increasing and converges to x. Briefly, in the construction used to realise any non-negative Π_1-computable real number as entropy of a multidimensional SFT, the machines are coded in order to control the density of some 'random bits' generating entropy.

These constructions have interested a relatively large community of researchers in mathematics and computer science, and several results have been published in this direction. In particular :

1. A characterization of the entropies of cellular automata [GZ12] in dimension two,
2. The simulation of effective subshifts by bidimensional subshifts of finite type [AS13],
3. The characterization of the possible sets of periods [VJ15].

Most of these results have been reproduced with another method, using fixed-point tilings [DR12]. Similar methods have been also used independently in quantum physics to prove the undecidability of the spectrum gap [CPGW'] (this result has interested physicists for the reason that many of the properties of physical systems are related to the spectral gap).

1.3 Effect of dynamical restrictions

Motivated by the development of efficient methods to compute the entropy of some multidimensional subshifts of finite type that appear in statistical physics, such as the hard core model [P12], R.Pavlov and M.Schraudner defined the property of block gluing, which is verified in particular by the hard core model, and consists in the possibility to assemble two block patterns in a configuration of the SFT, as soon as the distance between these two patterns is greater than a constant. They proved that block gluing multidimensional SFT have a computable entropy (where a number x is computable when there exists an algorithm which on input n outputs a rational r_n at distance lower than $@^{-n}$ from x), while for non-constrained multidimensional SFT it is not computable in general (consequence of Hochman and Meyerovitch's result). More broadly, other dynamical restrictions were known to have an effect on the values of some topological invariants : for instance, it was known (folklore) that the entropy of a minimal multidimensional SFT is equal to zero and that the entropy dimension of a k-dimensional SFT is at most $k-1$. The most recent results in this direction question the effect of dynamical restrictions on the computability of topological invariants (in particular entropy), through the possibility of encoding Turing computations in multidimensional subshifts of finite type that satisfy these restrictions. For instance, Hochman and Meyerovitch mentioned the possibility of characterizing the possible values of entropy for transitive multidimensional SFT as an open problem [HM10].

1.4 Statement of the general problem approached in this project

Since the uncompleteness and then undecidability results of Godel and Turing during the XXth century, it is known that there are limits to what can be computed, even algorithmically. Although very abstract, these results have been derived to answer more concrete mathematical questions, such as Hilbert's Xth problem :

Question 1 (Hilbert's Xth problem). *Does there exist an algorithm that decides if a diophantian equation has integer solutions or not ?*

In 1970, J.Matiiassevitch provided a negative answer to this problem, characterizing the sets of solutions of a diophantian equation as exactly the recursively enumerable sets (whose list of elements are the outputs of a Turing machine). Against this phenomenon, the natural reaction is to seek to understand what makes the domain to which mathematics were restricted thus far *computable*. More concretely, one will seek to understand the limit between computability and uncomputability in some restricted contexts.

In particular, it appears with the recent results presented above, that multidimensional symbolic dynamics (the study of multidimensional subshifts of finite type) are particularly well suited for the exploration of this general problem (for which answers are rare). My project is to study its particular instance of how dynamical restrictions affect the computability of entropy for dynamical systems, and in particular multidimensional subshifts of finite type.

1.5 Alreadty initiated approaches

Since my research project lies in continuity of past results I obtained with co-authors in this direction and depends on them, I shall first provide an abstract of these results.

1.6 Restriction to quantified block gluing multidimensional subshifts of finite type

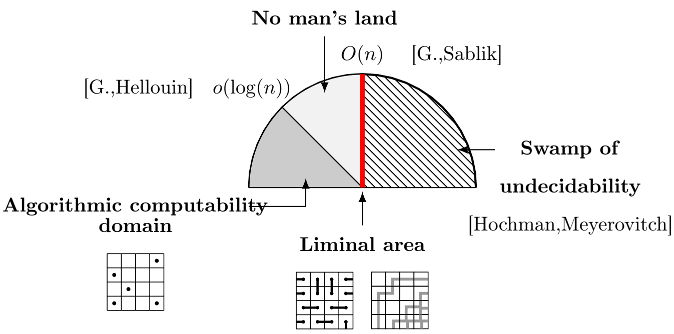
Several results have been obtained in the last decades on the computability of entropy for various dynamical systems, following a question by Milnor in 2002. These results often identify two regimes (computability and uncomputability) for entropy over various families of dynamical systems : for instance, piecewise linear maps of the interval [Koiran01], Turing machines [Jeandel14] [DB04] and cellular automata [DMM03] [**?**].

Given the recent results obtained by Pavlov and Schraudner and Hochman Meyerovitch for multidimensional subshifts of finite type (corresponding to two computability and uncomputability regimes), in order to approach the limit problem, we defined a quantification of Pavlov and Schraudner's block gluing property using the minimal distance of gluing (which depends on the size of the block patterns). According to this minimal distance function, the aim was to characterize precisely a "threshold" for this quantification.

More formally, we define the quantified block gluing property as follows :

Définition 2. *For a function $f : \mathbb{N} \to \mathbb{N}$, a SFT is said to be f-block gluing when for all pair of size n block patterns that appear in a configuration of the SFT can be glued and completed into a configuration of the SFT, whenever the distance between the blocks is greater than $f(n)$. A SFT is said to be linear block gluing when it is f-block gluing for a function f such that there exists a $C > 0$ such that for all $n \geq 1$, $n/C \leq f(n) \leq Cn$.*

We identified three zones, illustrated on the following diagram : on the left one the entropy is computable (a consequence of a later theorem proved with B.Hellouin) and on the right one the entropy is uncomputable (consequence of a characterization of the possible values of entropy as the Π_1-computable numbers). In the middle zone, we have no information, even on the existence of bidimensional SFT. The diagram also shows some examples of bidimensional SFT in relation with these zones.



The main theorem that we proved is the following :

Théorème 2 ([GS17a])**.** *The possible values of entropy for linear block gluing bidimensional SFT are the Π_1-computable non-negative real numbers. Moreover, a bidimensional SFT which is f-block gluing with $f(n) = o(\log(n))$ has a computable entropy.*

As the linear block gluing implies transitivity, this theorem provides an answer to the open problem of Hochman and Meyerovitch [HM10], introduced above. For this work we developped several tools that we refined further in order to provide constructions of minimal multidimensional SFT implementing computations of arbitrary Turing machines (a folklore problem that was considered very hard) that helped in particular to characterize the possible values of entropy dimensions for tridimensional SFT, as follows :

Théorème 3 ([GS17b])**.** *The possible values of entropy dimension for minimal tridimensional SFT are the Δ_2-computable numbers in $[0, 2]$.*

As we understand quite well the two computability and uncomputability zones, my research project in this direction consists in understanding the transition zone, through specific questions that I expose in the following. These questions can be of interest for researchers in combinatorics.

1.7 The characterization of a threshold for decidable subshifts

In order to go further in supressing this gap between the computability zone and the uncomputability one, another strategy is to study a similar problem for a class of dynamical systems that is close enough to multidimensional subshifts of finite type, but also more flexible. With B.Hellouin, we chose to study this problem for decidable subshifts, which are the subsets of some $\mathcal{A}^{\mathbb{Z}^k}$ such that there is an algorithm which decides if a pattern appears in a configuration of the subshift or not. We arrived at a precise characterization of a threshold :

Théorème 4 ([GH18]). *Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. If $\sum_n \frac{f(n)}{n^2}$ converges towards a computable number, then there exists an algorithm which computes (uniformly) the entropy of the d-dimensional decidable and f-block gluing subshifts. If $\sum_n \frac{f(n)}{n^2} = +\infty$, the possible values of the entropy for d-dimensional decidable and f-block gluing subshifts is the set of Π_1-computable non-negative real numbers.*

There is still some work in this direction, as a small gap remains : we still don't know what happens when the sum $\sum_{n=1}^{+\infty} \frac{f(n)}{n^2}$ is a real number but is not computable.

1.8 Generalisation of the approach

Despite the fact that recent results make multidimensional subshifts particularly suitable to approach the limit problem, we showed, with A.Herrera, M.Sablik and C.Rojas, that it is possible to arrive at significant results on the effect of some dynamical restrictions on computability of entropy for other classes of dynamical systems, in particular on the interval. This leaves some hope for other computational threshold characterizations for entropy, and a better understanding of this phenomenon in general, with a possible application to multidimensional subshifts of finite type.

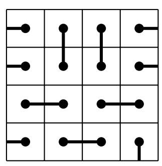
We constructed a general frame for a generalisation of this problem to dynamical systems on compact topological spaces [GHSR20]. In particular, using computable analysis [Weihrauch], we defined a notion of computable dynamical system, and proved that the entropy of a computable dynamical system is Σ_2-computable in general. We obtained some characterizations of possible values of entropy over some families of systems, such as surjective computable Cantor maps (Σ_2-computable non-negative numbers), computable interval maps (Σ_1-computable non-negative numbers), making use of the notion of horseshoe. The work of A. Herrera led also to a result on the uniform computability of entropy on the family of quadratic maps on the interval.

Two directions are possible for further research : the study of particular families of systems on the interval, and general characterisations results for other topological spaces, such as surfaces.

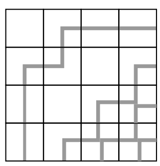
1.9 Computations of entropy in statistical and quantum physics

In parallel to the lines of research in computability theory presented above, physicists in statistical and quantum physics developped, all along the XXth century, some tools in order to compute the entropy for models of some physical phenomena. For some of these models (known as exactly solvable lattice models), that can be seen as multidimensional subshifts of finite type, physicists computed an exact formula for the entropy. Interestingly enough, these models lie on a "liminal area" between the transition zone and the uncomputability zone identified in the work on the block gluing property. A better understanding of the computation method could provide us with a better understanding of the transition, through the generalisation to families of multi-dimensional subshifts of finite type. In particular, we would like to understand what are the properties of these SFT that make their entropy computable.

The most simple one is the dimer model, whose configurations consist in coupling neighbour positions of $\mathbb{Z}^2$, as on the following picture :



Its entropy has been computed rigorously by P.W.Kasteleyn [K61] using a method that is very specific to this model, seeing the number of size n patterns that appear in a configuration of the SFT as the Pfaffien of a certain matrix, which can be computed with linear algebra methods. The second model of interest is the square ice which is isomorphic with the SFT whose typical patterns look like :



Its entropy has been computed by E.H.Lieb [L67], whose method relies on a convergence hypothesis left without proof. In order to have a better understanding of the reasons why this method works, I have decided to search for a fully rigorous derivation of the entropy value. A lot of work has been necessary for me to gather the various arguments used in the proof, connecting them with new results, in particular by K. Kozlowski [K15], and complete the several holes still left. The final result was the following theorem :

Théorème 5. *The entropy of square ice is equal to* $\frac{3}{2}\log_2(4/3)$.

Since some of the elements of the proof of this theorem can be defined (for instance solutions of Yang-Baxter equations) and studied for any multidimensional SFT, one could study these notions for some relatively simple SFT that are well known to theoretical computer scientists and stayed out of the scope of physicists. In particular, one could search to derive bounds on the complexity functions.

On the other hand, one cannot hope to apply directly this method to large classes of multidimensional SFT. A possible direct line of research consists in searching for rigorous proofs of computations for the few other models that were considered by physicists, that are far from being rigorous in their current state. Ultimately, I would like to develop more combinatorial methods to compute the entropy of multidimensional SFT.

2 Research project : frontiers of undecidability in symbolic dynamics

In this section I will expose, when possible, more precise questions in the directions of research described in the last section.

2.1 On the transition zone for multidimensional subshifts of finite type

2.1.1 State of the problem

With Mathieu Sablik, we determined a transition zone for the computability of entropy of multidimensional SFT using a quantification of the block gluing constraint. Our main conjecture is the following :

Conjecture 1. *The entropy of a* f*-block gluing bidimensional SFT such that* $f(n) = o(n)$ *has a calculable entropy.*

Most of the tools used to explore the transition (characterization of the entropies with constructions embedding Turing machines and combinatorial arguments) can not be used directly in this zone. We need thus to develop a new strategy and new tools to explore this zone.

Construction of examples of SFT in the transition zone In a shorter term, I would like to answer the following intermediate questions :

Question 2. *Does there exist a bidimensional SFT in the transition zone ?*

Question 3. *Does there exist such a SFT whose language is undecidable ? (In other words, is there an algorithm that decides uniformly over the transition zone if a pattern appears in a configuration of the SFT ?)*

These two questions are a priori intertwined : indeed, we know how to construct a bidimensional SFT which is linearly block gluing and whose language is undecidable. The construction relies on the implementation of universal computation over triangular areas that grow undefinitely in the vertical direction and whose rows can be shifted arbitrarily one from the other in any of the two horizontal directions. If one could build areas that grow with lower speed such that the resulting SFT would be inside the transition zone, one could hope to make a similar construction in order to answer the second question, with some work to adapt the construction.

Some natural ideas seem to lead to constructions of bidimensional SFT with gluing function $f : n \mapsto \sqrt{n}$ and f is the reciprocal of $n \mapsto n\log(n)$ (using the implementation of counters on the rows of the growing areas, that regulate their growth). The first step in this research axis would be to finalize these constructions. The second step would be to generalize the construction, and eventually answer (even partially) the following question :

Question 4. *What are the possible gluing functions in the transition zone ?*

However, we don't know how far one can push this generalization. In the construction using counters to regulate the growth of the areas, the counter has to satisfy a condition on the direction of propagation of the information. I would like to formalize this condition, and realize all the block gluing functions resulting from it. In a third step, I would like to understand what are the functions that satisfy this condition.

Families of bidimensional SFT In order to explore the transition area, another idea is to study particular (and sufficiently large) families of bidimensional SFT. The first family that I have in mind is constituted with generalizations of the dominating sets SFT (that we studied with A.Talon in [TG19]), which is the set of configurations on the alphabet $\{0,1\}$ such that every position in $\mathbb{Z}^2$ is dominated, meaning that there is another position in its neighbourhood whose symbol is 1. Some generalizations are already studied in graph theory : minimal and total dominating sets for instance. More generally, one can augment the alphabet, the neighbourhood, and the number of dominant and dominated positions forced in the neighbourhood for each position. The second family is the one of homshifts, that consists in subshifts whose alphabet is the vertex set of some finite non-directed graph, and whose configurations are such that the symbols on two neighbours are neighbours in this graph.

For the first family, most of the simplest examples are either not block gluing, linearly block gluing, or block gluing (with a constant gluing function). I conjecture that this is the case for all the subshifts in this family. In order to approach this conjecture, the question that I would like to answer in this direction is the following :

Question 5. *Is there an algorithm that decides, given an element in the dominating sets family of bidimensional subshifts of finite type, if this algorithm is linearly block gluing ? Constantly block gluing ?*

In the second family, most of the subshifts do not satisfy a block gluing property, but are all "linearly transitive", some "constantly transitive". For this family, I would like to define an operator that transforms these SFT into linearly or constantly block gluing ones, and then ask a similar question as for the first family.

Decidable subshifts With B.Hellouin, we characterized a precise threshold for the computability of entropy for decidable subshifts :

Théorème 6 ([GH18])**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. If $\sum_n \frac{f(n)}{n^2}$ converges towards a computable number, then there exists an algorithm which computes (uniformly) the entropy of the d-dimensional decidable and f-block gluing subshifts. If $\sum_n \frac{f(n)}{n^2} = +\infty$, the possible values of the entropy for d-dimensional decidable and f-block gluing subshifts is the set of Π_1-computable non-negative real numbers.*

Although this characterization is very precise, some gap remains : we don't know if the entropy is computable when the series converges to a number that is not computable. The difficulty here lies in the fact that both arguments for computability and uncomputability break completely in this case. In order to fill this gap, the direction that I would like to take (still at an exploratory stage) is to study in a more precise way the mode of convergence of the $\sum_k \frac{f(2^k)}{2^k}$ and the mode of convergence of the entropy, in particular to see if one can impose more constraint on the convergence to the entropy in the uncomputability regime. Although this question seems quite hard, its any progress in this direction would be highly significant in our understanding of the computational threshold phenomenon.

Characterization of entropy values below the threshold In the same vein, I would like to mention some hard questions whose answer would be very valuable for my problem. These are variants of the same question, asked by Pavlov and Schraudner, about the characterization of the possible values of entropy for block gluing bidimensional subshifts of finite type :

Question 6. *What are the possible values of entropy for bidimensional block gluing subshifts of finite type ? For f-block gluing bidimensional SFT such that $f(n) = o(\log(n))$? For decidable subshifts that are f-block gluing with f such that the series $\sum_k f(2^k)/2^k$ converges towards a computable number ?*

This question, which can also be formulated for other dynamical restrictions such as strong irreducibility [HM10],was answered partially by Pavlov and Schraudner, who realized a sub-class of the class of computable numbers defined by a condition on the computation speed as entropies of some block gluing bidimensional SFT. Their construction is based on a natural operator that transforms any SFT into a block gluing one. and preserves entropy as long as the initial SFT, denoted X, satisfies a condition on the convergence speed of the sequence $\log(N_n(X))/n^2$ towards $h(X)$. It is then sufficient to realise any number in the class defined above as the entropy of a bidimensional SFT that satisfies this condition. Their construction relies then on implementing Turing machines in order to constrain an infinite word to be a pseudo-sturmian (the properties of these words imply the desired property) that generates entropy.

In order to go further, we need to understand better the relation between the computation speed of the algorithm that computes a number and the convergence of $\log(N_n(X))/n^2$ for bidimensional subshifts X whose entropy is this number. Also, given the extreme complexity of this type of construction, a possible strategy could be to attempt to answer this question first in high dimension using the simplest possible construction as a basis in order to understand the relation mentioned above (for instance allocating one bit by dimensional section in which all the machines act only on this bit).

2.1.2 Other research directions

I describe here other research directions and reflexions that are more broadly related to the theme of the first section. An important part of my reflexion was around the following conjecture :

Conjecture 2 (B. Weiss)**.** *Any sofic subshift has a SFT cover that has same entropy.*

Here a sofic subshift is a subset of some $\mathcal{A}^{\mathbb{Z}^2}$ such that there exists a SFT and a local function from this SFT to this subset that is onto. Such a SFT is called a cover of the sofic subshift. This conjecture is interesting for me since in order to answer it one would have to find new ways to build multidimensional SFT having properties related to entropy.

My approach of the conjecture would be based on the idea that one can build a sofic subshift without any SFT cover having the same entropy, by programming a SFT whose configurations are constituted with "objects" (such as curves, or rectangular areas) (this intuition, amongst others, is one reason for the study presented in Section 3) that are the support of some information transport whose aim would be to synchronise some bits of information, supported for instance by the extreme parts of these objects. The idea is that this information transport would be forced to have, in its definition, some orientation. Since the "objects" themselves would not have any specific orientation, these two orientations would be possible for the information transport. The sofic subshift obtained by "forgetting" the symbols coding for this orientation would be a good counter-example candidate for the conjecture.

We would then have to understand the relation between the entropy of a subshift composed with "objects" and a subshift enriched from this one with random symbols superimposed on specific parts of these objects, in such a way that we can prove the difference or equality of entropies of these two subshifts. This general question can be formulated apart from the conjecture.

I would also like to mention two very hard questions related to SFT in high dimension, whose solution can be highly valuable to understand the effect of dynamical constraints on entropy.

Question 7. *Does there exist a block gluing tridimensional SFT whose entropy is not computable ? Whose language is undecidable ?*

Question 8. *Does there exist an aperiodic block gluing tridimensional SFT ?*

Other physical quantities : the exemple of the spectral gap Recently, another undecidability result was proved for another physical quantity that can be defined for multidimensional SFT : the spectral gap. In order to explain this notion, I have to recall the notion of adjacency matrix : given a bidimensional SFT, its order n adjacency matrix describes which pairs of rows of n symbols can appear adjacent in a configuration of the SFT. A SFT is said to have a spectral gap when the difference between the maximal eigenvalue of this matrix and the second largest stays above a positive constante for an infinity of integers n. The result is the following :

Théorème 7 ([CPGW])**.** *It is impossible to decide if a bidimensional SFT has a spectral gap or not.*

This result was presented in an article published in the journal *Nature* [CPGW'], in a short version. The original article is very long and although the proof relies on Robinson construction, using quantum Turing machines instead of classical ones, it seems that this result was obtained independently from the recent development around Robinson construction led by M.Hochmand and T.Meyerovitch [HM10] or the fixed-point construction by B.Durand and A.Romashchenko [DR17]. Given that this type of constructions consists mainly in the introduction of new gadgets in the initial construction by Robinson, it would be interesting to better understand this construction in detail, extracting in particular new tools in order to approach questions related to computability of entropy.

One could also transport the theme of the effect of dynamical constraints on computability or decidability of the spectral gap :

Question 9. *Does there exist some dynamical constraint on bidimensional SFT that imply the decidability of the spectral gap problem ?*

Since numerous properties of lattice models are determined by the existence of a spectral gap, this problem is important for physicists [K17], and any constraint of this type could be interpreted as the localisation of what is computable in the world of lattice models, with theoretical and practical implications.

A correspondance between dynamical constraints and topological invariants In another work with M. Sablik, we proved the following result :

Théorème 8 ([GS17b]). *The possible values of entropy dimension for minimal tridimensional SFT are the Δ_2-computable numbers in* $[0, 2]$.

Given this result, one can envision two directions to follow :

1. Introduce a quantification of the minimality property with a minimal distance separating two occurences of the same pattern in a configuration (called quasiperiodicity function), and ask about the effect of this distance on the possible values of entropy dimension for multidimensional SFT.
2. It is remarkable that the entropy of a minimal SFT is zero, while the only number that is the entropy dimension of a linear block gluing tridimensional SFT for a gluing function bounded from below by $n \mapsto Kn$ for any K is 3. As a consequence, there is a correspondance between two topological invariants and two dynamical constraints, since the possible values of any of these two invariants for tridimensional SFT under the corresponding dynamical constraint form a non-trivial set, while this set is trivial when the constraints are interchanged. One can imagine a larger correspondance between dynamical constraints similar to the linear block gluing and minimality, defined for instance with logical propositions on patterns, and some topological invariants such as entropy and entropy dimension, by variation of the functions implied in the definition of the invariant. According to a common reflexion with I. Torma, it is possible to characterize precisely a threshold for bidimensional SFT for polynomial complexity [M11] and a notion of linear transitivity. Such a correspondance could lead to a threshold on the function defining the topological invariant at which the threshold of the invariant is transformed into larger transition area. This could help to understand better the transition area for entropy and linear block gluing.

2.2 Exact computation of entropy for some multidimensional SFT

Multidimensional subshifts of finite type, known as particular lattice models in physics, have been objects of interest for physicists since the beginning of the XXth century, beginning with the Ising model and its solution by L.Onsager. Here the word solution refers to the possibility to compute formally (although not necessarily in a rigorous way) some global quantities related to the model, such as entropy (or free energy). Although entropy computation were considered as a pretext for the development of tools in order to compute more refined quantities such as correlation functions, a lot of effort has been deployed to compute the entropy for some of these models. However, the litterature on the subject is only partially rigorous and quite obscure and unaccessible to mathematicians. My aim is to find rigorous methods by other means (in particular combinatorial ones, instead of heavy functional analysis) and complete existing computation methods into rigorous ones when possible, in order to construct a general frame for the extension of these methods to families of multidimensional subshifts of finite type, thus approaching the limit between computability and uncomputability "from below".

2.2.1 Rigorisation of exact entropy computations

Besides the models of dimers and square ice, there exist two models for which exist some entropy computations : the eight vertex and the hard hexagons.

The first one was solved by R.Baxter [Baxter], and the computation method can be decomposed, as in the case of square ice, into the computation of a formula for the largest eigenvalue of adjacency matrices, and then the asymptotic analysis of this sequence of eigenvalues. After some work with P.Melloti, we concluded that this computation has two problems : the first one is that the first part of the method is not accessible to mathematicians and that it is difficult to evaluate its correctness, and some arguments are missing for the second part. Hence the following research direction :

Research direction 1. *Review the literature and synthetise a rigorous solution of the eight-vertex model. Adapt then the tools developped for the second part in the case of square ice to the eight-vertex model.*

The hard hexagons case is more difficult, since the computation of its entropy by R.Baxter [Baxter] is clearly non rigorous and there is no visible possibility to make it rigorous in reasonable time. This computation relies on a functional equation satisfied by a path of diagonal transfer matrices, for each of the two diagonal directions. With the help of some particular properties of these paths of matrices, these functional equations are transported to the largest eigenvalues of these matrices. Then the value of entropy is derived from a hypothesis of equality between the two largest eigenvalues of the two diagonal transfer matrices for the same size and the same parameter. The consequence of this hypothesis is that the entropy satisfies a system of functional equations which is sufficient to determine it as a function of the parameter. Given the nature of this hypothesis, one can consider the following research direction :

Research direction 2. *For which multidimensional SFT is it possible to indentify the two largest eigenvalues of the two diagonal adjacency matrices ?*

One can note that there are some important models for which no formula is even conjectured : for instance the hard core model, and non-attacking kings model. In order to approach these models, I would like to attempt finding approximate solutions of Yang-Baxter equations on higher block representations, possibly by computer means. There is also a part of the literature in physics around some solvable models produced by abstract solutions of Yang-Baxter equation and related to properties of algebraic structures, such as braid algebras. We don't know if these results are completely rigorous and if entropy is even computed for these models. In any case, it would be important to consider this literature for the question of computability of entropy for multidimensional SFT. This can be extended also to geometric tilings solutions, such as tilings by squares and triangles [K99], or octogonal tilings [GN96].

2.2.2 A combinatorial approach to entropy computations

Besides the rigorisation of existing computations, I would like to develop a more combinatorial approach, that would aim in particular at developping combinatorial methods for the computation of entropy. I see the following various strategies :

Extending existing combinatorial methods : In our work with M.Sablik [GS17a], we used in the proof of the main theorem an exact computation for the entropy of a family of bidimensional SFT obtained by some modifications of the square ice SFT (by compactification of the curves). While the proof is not trivial, it uses only combinatorial arguments. The strategy is to extend this proof to other subshifts of finite type obtained by other modifications of this SFT that are simple and close enough to the initial one, or by additional modifications, in a way that the method is still valid in its principle but has to be adapted in a non-trivial way.

Exploration of classes of bidimensional SFT with fixed entropy In order to understand better the case of square ice (and eventually find a computation method that can be extended to a large enough class of bidimensional SFT), I would like to search for a combinatorial method for computing its entropy :

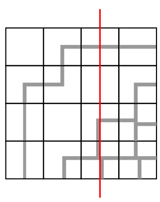
Question 10. *Does there exist a combinatorial proof for the value of square ice entropy ?*

There are two possible way to approach this question. First, one can use the arguments used by Baxter for the hard hexagons on square ice, and take advantage of the existing rigorous proof to make a comparison and find some significant relations that could help refine Baxter's argumentation. Second, one can try to realise the entropy of square ice $\frac{3}{2}\log_2(4/3)$, simplifying the general construction by Hochman and Meyerovitch for this special case, in such a way that the construction appears close enough to the square ice SFT (one way to do it is to note first that the $(4/3)^n$ can be expressed as a binomial sum, then try to realise individually each term of the sum as the cardinal of a set of patterns, by selecting a subpart of the set of patterns on the alphabet $\{0,1\}$ such that the number of 1 symbols is exactly some integer k). The next step would be to search for a natural isomorphism between the constructed subshift and square ice, leading to a new proof of the value of square ice entropy.

Realising complex entropies with operators In [GS17a], we use a sequence of distortion operators $(T_r)_r$ on bidimensional subshifts of finite type, acting on entropy by adding the term $\log_2(r+1)/r$. In order to realise more "complex" entropies based on existing constructions and examples, one could ask if there exist other operators that act on entropy as a pre-fixed function, for instance :

Question 11. *Does there exist a bidimensional SFT that transforms the entropy of a SFT into its square ?*

Understanding subshifts of square ice through decidability questions Also in [GS17a] , we use subshifts of square ice defined by imposing some constraints on vertical sections of configurations to belong to a unidimensional SFT on alphabet $\{0,1\}$, where a vertical section of a configuration is a biinfinite word obtained by considering the restriction of the configuration to a column, writing 1 on a position if it contains a segment of curve and otherwise 0. The following figure illustrate this meaning of the word section :



As an intermediate question towards the one of computing entropy on this class of subshifts, one can question the relation between the unidimensional SFT, denoted X here, and the bidimensional SFT generated, denoted $s(X)$. For instance :

Question 12. *Does there exist a unidimensional SFT X whose entropy is positive such that the entropy of $s(X)$ is zero ? And conversely ? Is it possible to decide algorithmically in X if the entropy of $s(X)$ is positive ?*

This research direction could benefit new advances made by J.Esnay and M.Sablik on bidimensional subshifts of finite type defined by vertical and horizontal constraints.

3 Towards a measure of 'organisedness' for dynamical systems

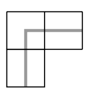
In its current state, G.Tononi's (still controversial) theory defines a quantity that is posited to measure the degree of (phenomenal) consciousness of a dynamical system (in other words how much it experiences things). A central idea of its construction is to 'formalise' the existence of a dynamical system for its environment by the causal power it has on it, as a whole and not as a collection of separate elements. Consciousness is then related to the existence of a dynamical system for itself, and thus the causal power it has on itself. The definition of this quantity follows multiple past attempts to formalise the specific complexity of the human brain, such as the neuronal complexity [TSE] or the functional clustering index [TMRE] which mathematicians coming from dynamical systems community have started to study [BZ12][PW18]. In a broader scope this theory introduces a way to think about information in terms of causality, coming back to the ethymology of the term in-formation. Despite the interest of the mathematical community in this theory, this has been limited to litteral transcription in other formalisms such as category theory [TTS16], dynamical systems defined by differencial equations [EGLPS], processes physics [KT20], quantum physics [ZTC], information geometry [OTA]. After analysing this theory for some time, I concluded that it has a number of severe defects, in particular the inability of the formalism to extend beyond the domain on which it is intuitive, as well as the untractability of the quantities it defines. I attribute these defects to the fact that the theory is turned more towards a methaphysical use of mathematics than mathematics themselves. Despite this fact, I realised during this time that causality notions could be used in order to understand 'visual patterns' that we naturally distinguish in configurations of multidimensional subshifts of finite type and how these patterns interact, sometimes in an *organised* way, and thus that I could use causality to propose a definition of organisedness. In the long run this approach could be used also to think about information flow in dynamical systems, and attack fundamental problems in multidimensional dynamics.

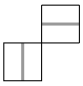
3.0.1 The idea of causal structure

For this purpose I had to adapt causality notions, used in the framework of integrated information theory, to the context of multidimensional subshifts of finite type, in particular in combinatorial terms instead of statistical ones.

Causal relations and irreducibility. I won't explain in detail J.Pearl [Pearl]'s work but rather the understanding I got of mathematics of causality through integrated information theory. Let us consider a finite probabilistic dynamical system, meaning that it is composed of a finite number of units whose possible states are described by a finite set and a transfer matrix which gives the probability of each unit's state at time t+1 knowing the system's state at time t. We say that a unit in a certain state at time t has causal power on another unit when, knowing the state of the first unit at time t, the probability distribution over possible states of the second unit at time t is not uniform. In other words, the state of the first unit at time t affects the one of the second unit at time t+1. When this is the case there exists an (oriented) causal link between these two units. This notion can be naturally extended to groups of units. Moreover it is possible to quantify the causal power with the distance between the uniform probability measure. How much this causal power is affected by cuts into two parts of a group measures how much a group has causal power beyond its partitions.

Adaptation to multidimensional subshifts of finite type. These notions can naturally be adapted to multidimensional subshifts of finite type : given a multidimensional SFT and one of its configurations, a symbol on a certain position has a causal link with another position when the set of possible symbols on this second position is restricted by the presence of the symbol on the first position. For instance for the square ice SFT, as a consequence of the local rules defining this SFT, in any configuration in which the following pattern appears



the pattern  causes .

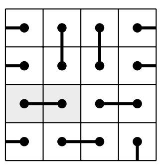
Another interpretation of information. In the latest developments of G.Tononi's approach, causality is explicitely related to information, and this relation is an outcome of the intuitive need to go beyond information formalism derived from C.Shannon's ideas, that are not adapted to understand exchanges of information in natural science. In the framework of multidimensional subshifts of finite type one can interpret a causal link, defined above, as a transfer of information (ethymologically an in-formation) in the sense that the second unit is in-formed by the state of the first one. One can also say that the second unit has, by its state, an "information" on the first one's state (one can note that this vocabulary is already used, with good reason, by A.Romaschenko and B.Durand to describe in an intuitive way their constructions [DR17]). It seems to me that this notion of information is more adapted than C.Shannon's one to interpret transfers of information "inside" dynamical systems such as multidimensional subshifts of finite type. Moreover given that this paradigm shift has been fruitful for the development of G.Tononi's theory, it is relevant to explore this point of view in this direction also, in particular by interpreting or adapting existing notions related to transport of information, intuitively or explicitely, in terms of causality. In these notions one can include for instance sofic subshifts or communication complexity (as well as Lyapunov exponents for cellular automata or determinism or expansivity directions). Moreover I am interested in some open questions related to transport of information, amongst which the following ones : i) caracterizing bidimensional sofic subshifts, which intuitively are the ones that satisfy a bound on the quantity of information transported locally ii) caracterize unidimensional subshifts such that the bidimensionnel subshift which consists in a free "pile" of its configurations is sofic.

Causal structure. In integrated information theory, the causal account of a finite probabilistic dynamical system is the collection of all the causal links between sets of units together with the causal power that correspond to each link. In its most recent developments, the theory refined progressively this notion in order to relate brain architecture to direct observation (and theoretise consciousness) by defining a structure for this causal account : the fundamental identity of this theory is then the identity between the causal structure of a system and what it experiences (if it experiences anything at all). Roughly the structure of the causal account consists in a 'nucleus' of causal links from which can be derived the whole causal account by certain operations (just as a group can be recovered by products of its generators with its inner operation). For instance one can consider to be in this 'nucleus' only the irreducible causal links, meaning the ones which can not be decomposed in two independent causal links (more precisely, a causal link between two sets of units is reducible when there exist partitions of these two sets, on of which is non trivial, such that the first (resp. second) sub-set of unit of the first set causes the first (resp. second) sub-set of the second). One can also consider to be in the nucleus any causal link between two sets such that if a third set is caused by the first and causes the second, then this third one also causes the first.

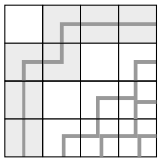
3.0.2 Towards the 'right' mathematical definition of causal structure

I have shown in an unpublished paper (*A formal window on phenomenal objectness*) that one can use variants of this notion of causal structure, in particular adapted to the framework of multidimensional subshifts of finite type, in order to derive, for some simple subshifts of finite type (such as dimers and square ice), certain particular patterns, that I would call 'objects', which one naturally distinguishes when one manipulates this systems, for the reason that any configuration is composed of them. In a sense the causal structure of the SFT reveals how each of its configurations is *organised.*

For instance, in a configuration of dimers such as the following one, one naturally distinguishes patterns such as the one highlighted with light gray :



which give their their name to the model. We tend to consider them as 'things' (which corresponds to the notion of complex in integrated information theory) that are contained in the configuration, thought as a visual experience. Any configuration is thought as an 'assembling' of dimers, meaning that one can describe this configuration with the positions and orientations of these objects. This description reflects the degree of 'organisedness' of the configuration (here its relative 'unorganisedness'). There are similar intuitions for all the multidimensional SFT that specialists manipulate, such as the square ice model, in whose configurations one distinguish naturally the 'curves' such as the one highlighted in the following pattern :



In a preprint [G20] that one can find here, I defined a formal framework in order to study this matter, and proposed a pair of ways to use causality to account for the distinction of certain patterns in simple bidimensional subshifts of finite type. To my knowledge there is no other way than causality to formalise this intuition in a relative uniform way (that would be the same for multiple SFT), and in these terms, these patterns can be seen as orbits in the causal structure, considered as an oriented (infinite) graph, which satisfy certain conditions. Contrarily to integrated information theory, I don't want to posit a particular formalisation based on the intuition only, but to leave open the multiplicity of possible definitions, selecting some with the criterion of uniformity : the more SFT such a formalisation is valid on, the more it is significant. Let us notice that there are particular subshifts of finite type (for instance the space-time diagrams of some cellular automata, such as J.Conway's 'game of life') where this distinction of certain patterns (the 'gliders' for instance) is particularly visible (and of interest). However the study of their causal structure is highly non trivial. In the long run I whish to be able to related these patterns to the causal structure of this SFT. In order to do this the strategy would be to focus first on 'causally simple' bidimensional subshifts of finite type, with increasing complexity. Here the interest of the multidimensional subshifts of finite type class of system is that it contains multiple relatively simple SFT relatively to causality, yet sufficiently complex to restrict the range of possible formalisations. With this restriction one would be able to bring the 'right' notion of causal structure to analyse cellular automata such as the game of life.

Once formalised this notion of 'object', my intuition is that the degree of 'organisedness' of a dynamical system could be expressed in terms of an 'objectal complexity', meaning the complexity of the set of possible objects in configurations of the SFT (their number for instance) and the complexity of the ways they 'interact' (forcing each other's orientation, overlapping or not, etc). Let me notice that a proper notion of organisedness would in return be of interest for the quantification of 'consciousness' (the more one system is able to distinguish things in its environment, the more it is conscious), but also (as mentioned above) to provide a notion of information that can be useful in numerous fundamental mathematical questions involving information flow in dynamical systems.

Other criteria. In the selection of the 'right' notion of causal structure, one could use also less intuitive and more mathematical criteria, such as the possibility to answer to the following question :

Question 13. *Is it possible to characterise the set of possible causal structures for unidimensional SFT ? Of finite non-deterministic dynamical systems ?*

Here I restrict to unidimensional SFT for the fact that it is easier to answer classification questions (such as possible entropies) for these systems. In this direction, I suspect that an understanding of these causal structures could be helpful in attacking some hard open problems such as the classification of conjugation classes or the decidability of the conjugation problem (see for instance this list of open problems). One could use also as a criterion the question of the relation between a dynamical system and its causal structure : for which notions of causal structure is there a one-to-one function between the system and its causal structure ? In particular for some simple ways to think about causal structure, one can observe the existence of such a function in general or the non-existence in general. It is possible that a 'limit' notion between these would be significant.

3.0.3 Machine learning with neural networks

The direction of research exposed in the last section is also motivated by the question of explaining the striking recent advances of machine learning and the efficiency of neural networks in the execution of various tasks, amongst which the detecting of certains 'objects' in pictures : for instance some competition has been proposed on the platform Kaggle in 2013 which consisted in progamming efficient algorithms that would detecting, given a (numerical) picture, if this picture contains a dog or a cat, where the pictures belong to a fixed dataset. The contest was won by Pierre Sermanet with detection accuracy greater than 0.98 using deep learning.

For this reason machine learning (and in particular deep learning) attracted many researchers in the last decade, sometimes approaching it in a theoretical way : for instance see [Scholkopf] for an approach of machine learning based on causality. The physicist M.Mézard also exposed some ideas in a talk at the fourth workshop *Machine Learning for Physics and the Physics of Learning* (2019), that the efficiency of deep learning algorithm should be related to the (geometrical) *structure* of the set of data, that he calls *hidden manifold* of the data set (and that he studied for the dataset MNIST).

The approach described in the last section is parallel to this line of thoughts and actually is a form of systematisation of these, by considering a class of datasets which consist in (infinite for simplification) pictures, defined by local rules, and searching for a relation between these local rules (intimately related to the geometrical structure of the set) and the 'patterns' that we tend to perceive in them, using notions derived from causality. I thus believe that the outcomes that would follow from the direction taken in last section may have an impact in understanding machine learning.

3.0.4 Return to consciousness

In the long run, an appropriate notion of causal structure constructed mathematically would be useful for the study of consciousness (through G.Tononi's theory). I would like to mention two interesting problems that can not yet be approached, given the current formalism's complexity. One can consider these two problems as informal criteria for a 'right' notion of causal structure.

Integration maxima problem : G.Tononi proposes a quantity which measures the degree of causal interconnectivity of a complex dynamical system. Intuitively the greater this quantity the more the system is "integrated", an accepted criterion for a physical mechanism to support consciousness. On the model of entropy, it would be interesting to answer the question of the maximal value for this quantity over a sufficiently large class of systems, as well as the systems that realize this maximum (for which an answer would reveal the relevance of the quantity).

Equivalence with perturbational point of view : there exists another quantity proposed in the same research are, called *perturbational complexity index (PCI)*, which has been defined to measure the neural system's specific complexity, and is based on the stereotypical character of responses of this system to small perturbations. An important question raised by G.Tononi is of the possibility that this quantity is 'identical' to the one mentioned above. Although I don't show it in this research statement, it is possible to adapt this quantity to the framework of multidimensional subshifts of finite type.

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